Barsotti-Chevalley Thm:

G a gp var $/k$, then there $\exists$ a (smallest) affine connected normal subgp N contained in G s.t. G/N is an abelian var.

1) Natation: k field

Scheme: of finite type over k

alg gp: gp object in the cat Sch/k

var: geo-reduced sep scheme $/k$

gp var: gp objext in cat of connected var$/k$

Rmk: ① For alg gps G, smooth connected $\Leftrightarrow$ geo-reduced

② Alg gp is separated

$\Rightarrow$ So gp var are just smooth connected alg gp

2) Basic Facts

alg gp G: k-algebras $\longrightarrow$ gps

① $H \subset G$ is normal $\Leftrightarrow$ $H(R)$ is normal in $G(R)$ for any k-alg R

② $1 \to N \to G \to Q \to 1$ is exact $\Leftrightarrow$ $1 \to N(R) \to G(R) \to Q(R)$ & $G \to Q$ is faithfully flat

$G(R') \dashrightarrow Q(R)$

R' is a faithfully flat R-alg

③ affine, smooth, connected ~~are~~ are preserved under quotients and extensions

i.e. $1 \to N \to G \to Q \to 1$.

If G has $-$ $\Rightarrow$ Q has this $-$

If N & G have $\equiv$ $\Rightarrow$ G has this $\equiv$

Prop: ① Take H, N two alg subgps of G, N is normal

$\Rightarrow HN$ alg gp of G. If H & N $\equiv$ $\Rightarrow HN \left\{ \equiv \right.$

$$\begin{array}{ccccccccc} 1 & \to & N & \to & HN & \to & HN/N & \to & 1 \\ & & & & & & \downarrow \wr & & \\ & & & & & & H/H\cap N & & \end{array}$$

② Every alg gp contains a largest smooth connected affine normal gp.

Def: ① A pseudo-abelian var is a gp var such that any affine normal subgp is trivial

② An abelian var is a complete gp var

Rmk: ① An ab var $\Rightarrow$ pseudo-abelian

aff + complete $\Rightarrow$ 1 pt.

Eg: ① Elliptic curve is an ab var.

② A pseudo-ab var over a perfect field k is abelian

* p odd prime k char $k = p$, $\exists$ some regular proj curve X/k

such Jacobian $\mathrm{Pic}^{\circ}_{X/k}$ is p-abelian but not abelian

LM: G alg gp. it can be written as an extension

$$1 \to N \to G \to Q \to 1$$

where Q is p-abelian, N is normal affine

Pf: N = largest normal affine subgp var, $Q = G/N$

$\square$

Prop: ① A rational map from normal var to complete var is def on an open sub set whose complement is of codim ≥ 2.

② A rational map from smooth var to gp var is define on an open subset whose complement is either empty or of pure codim 1

③ A rational map from smooth var V to ab var is def on the whole V.

④ A morphism from a gp var to ab var is a composite of a homo with a translation

⑤ A ab var $\boxed{x \mapsto x^{-1}}$ $\Rightarrow$ A is commutative

Rmk: p-abelians are commutative.

3) Rosenlicht Decomposition Thm (ab subvar of an alg gp has an almost-complement).

LM1: $G \times X \to X$ G alg gp, X connected scheme.

If there $\exists$ a fixed pt P, then G must be affine.

(Borel Fixed Point Thm, G affine solvable, X complete $\Rightarrow \exists$ a fixed pt).

Pf: $G \hookrightarrow GL(n)$. G fixes P, G acts on the local ring $\mathcal{O}_P$.

$\mathcal{O}_P/m_P^{n+1}$ it commutes with extension of base

G-action will be define for each k-alg R

$$G(R) \longrightarrow \mathrm{Aut}(R \otimes_k (\mathcal{O}_P/m_P^{n+1}))$$

$\Downarrow$

Natural in R $\boxed{\rho_n : G \to GL_{\mathcal{O}_P/m_P^{n+1}}}$

$H_n := \ker \rho_n$ for each n, $H_n \supseteq H_{n+1} \supseteq \cdots$

G is noetherian $\Rightarrow \exists\, n_0$ s.t. $H_{n_0+1} = H_{n_0+2} = \cdots$

$H = \bigcap_{n\geq 1} H_n$

We consider $X^H \underset{\text{closed}}{\subseteq} X$, $\mathcal{I} \subseteq X^H$ sheaf of ideals

$\Rightarrow \mathcal{I}\mathcal{O}_X \subseteq m_p^{n+1}$ for any $n \geq n_0$

$\Rightarrow \mathcal{I}\mathcal{O}_X \subseteq \bigcap_{n\geq n_0} m_p^{n+1} = 0$ Krull Intersection Thm

$\Rightarrow \mathcal{I}\mathcal{O}_X = 0 \Rightarrow \exists$ an open subset $U \subseteq X^H$

But $\boxed{X \text{ irr}}$, X^H closed $\Rightarrow X = X^H \Rightarrow H = e$

$\square$

<u>Cor</u>: G connected alg gp. e unit element, $\mathcal{O}_e$

G acts on itself by conjugation, then it defines a rep $G \to GL_{\mathcal{O}_e/m_e^{n+1}}$, if n is large enough, kernel of this rep is $Z(G)$

Pf: $G/Z \times G \to G$

$\square$

<u>LM2</u>: G connected alg gp. Every ab-subvar A is contained in the center of G: $Z(G)$

A is normal in G.

Pf: Take large n, $\rho_n : G \longrightarrow GL_{\mathcal{O}_e/m_e^{n+1}}$

$\rho_n(A)$ closed $\Rightarrow$ affine + complete $\Rightarrow$ trivial

$A \subset \ker \rho_n = Z(G)$ $\square$

<u>LM3</u>: G commutative gp var $/k$

Take $V \times G \to V$ a G-torsor (V is the same as G

but forget the unit/identity element)

Then there $\exists$ morphism $\phi: V \to G$ and integer n

such $\phi(v+g) = \phi(v) + ng$ for $v \in V, g \in G$.

Pf: ① If $V(k) \neq \emptyset$. Take $P \in V(k)$, there $\exists$ a G-equivariant iso: $\phi: V \to G$ $\quad P \mapsto e$, $\quad v \mapsto (v-P)$ $\quad (v-P)+P = e$

$$\phi(v+g) = v+g-P = v-P+g = \phi(v) + 1\cdot g$$

② In general, V alg var, $\exists P \in V$ s.t. $k(P)$ is a finite sep ext of k of deg n. $K :=$

$$k^{alg} \,|\, \tilde{K} \,|\, K = k(P) \,|\, k$$

Take $P_1, \ldots, P_n$ to be the k^{al}-points lying over P. If set $\tilde{K}$ to be Galois closure of K in k^{al}

$\Gamma = G(\tilde{K}/k) \Rightarrow P_i \in V(\tilde{K})$

We have morphism defined over $\tilde{K}$:

$$V_{\tilde{K}} \to G_{\tilde{K}}$$
$$v \mapsto \sum (v - P_i)$$

As this map is G-equi, take Γ-invariant

$$\phi: V \to G, \quad \phi(v+P) = \sum(v+P-P_i) = \sum(v-P_i+P) = \phi(v) + nP$$

$\square$

Thm (Rosenlicht Decomp Thm)

Let A be an ab subvar of a gp var G. There $\exists$ a normal alg gp N of G s.t. the map

$$A \times N \to G, \quad (a, n) \mapsto an$$

is faithfully flat

~~[illegible]~~ with finite kernel.

If k is perfect, N can be chosen to be smooth
. (Replace N by N_{red})

pf: A is ab $\Rightarrow$ A is normal $Q = G/A$

$\exists$ faithfully flat homo $\pi: G \to Q$, $\ker\pi = A$

As A is smooth, π has smooth fibres of constant dimension $\Rightarrow$ π smooth.

Take a generic fibre $V \to \operatorname{Spec} K$ of π, then can regard V as a A_K-torsor

$\phi: V \to A_K$, extend it to rational map over k

$$G \dashrightarrow Q \times A \dashrightarrow A$$

rational map $G \dashrightarrow A$ is defined on the whole G
gp var ab var

$\Rightarrow$ it's a morphism.

$$\phi'(g+a) = \boxed{\phi(g)} + na$$

$$\phi': A \xrightarrow{\times n} A.$$

($\ker\phi'$ normal alg gp of G)

But $A \subset Z(G) \Rightarrow A \times N \to G$ is a homomorphism
$(a, n) \mapsto an$

Kernel is $N \cap A \Rightarrow$ finite gp scheme.

$\square$

4) Thm5 Ab var are proj

(complete + quasi-proj = proj)

Pf: (Sketch) $A \hookrightarrow \mathbb{P}^N$

D separated pts and tangent vectors

$\Rightarrow$ Let $f_0 \cdots f_n$ a basis of $L(D) \cup \{0\}$

$a \longmapsto [f_0(a) : \cdots : f_n(a)]$

Take a finite family of irr divisors Z_i

$\bigcap_{i=1}^{n} Z_i = \emptyset$, $\bigcap T_e Z_i = \{0\}$

$D' = 3 \cdot D$, $|D'|$ will satisfy our requirement $\square$

Thm6: Every homoge space is quasi-proj

5) Thm7: (Rosenlicht Dichotomy Thm)

If $k = \bar{k}$, G gp var $/k$. Either G is complete

Or it contains an affine alg subgp of dim > 0.

Def: Ration action of gp var G on var X

$G \times X \dashrightarrow X$

① e unit, $x \in X$, $e.x$ is defined $\overset{e.x}{=} x$

② $g, h \in G$, $x \in X$, if $h.x$ is de $g.(h.x)$ is def

$\Rightarrow (gh).x$ is def & $g.(h.x) = (gh).x$

LM: G gp var, X var, $\varphi: G \times X \dashrightarrow X$ rational map

If $\exists$ dense open subvar $X_0 \subset X$, s.t.

$G \times X_0 \dashrightarrow X_0$ is regular action

Then φ is a rational action.

Pf: ① $e.x = x$

② Exercise $\square$

$\{X, Y$ irr

② Exercise $\square$

Lm 9: $\alpha: X \to Y$ a regular dominant map $\begin{cases} X, Y \text{ irr} \\ Y \text{ complete} \end{cases}$

$D \nearrow$ prime divisor $\quad \alpha|_D : D \to Y$ is not dominant

Then there $\exists$ a complete var Y' such that

there $\exists$ a birational regular map $\beta: Y' \to Y$

such that $\beta^{-1}(\alpha(D))$ is a divisor on Y'.

$$\begin{tikzcd} D \arrow[r, hook] \arrow[rr, bend left] & X \arrow[r, "\alpha"] & Y & \alpha(D) \\ & Y' \arrow[ur, "\beta"] & \beta^{-1}(\alpha(D)) & \end{tikzcd}$$

Pf: $k(Y) \hookrightarrow k(X)$. D, $\mathcal{O}_D$ is a DVR.

$v: k(X)$, $\quad w = v|_{k(Y)}$ it's nontrivial

$\uparrow$ DV $\quad$ is a dv on $k(Y)$

$\mathcal{O}_w$ valuation ring, $k(v)$ $\quad k(w)$

$$\mathrm{tr.deg}_k k(w) = \mathrm{tr.deg}_k k(v) - \mathrm{tr.deg}_{k(w)} k(v) \qquad \dim Y = n.$$

$$\geq (\dim X - 1) - (\dim X - \dim Y)$$

$$= n - 1$$

$f_1, \ldots, f_{n-1} \in \mathcal{O}_w$ their images in $k(w)$ alg independent $/k$

Consider "proj emb" $\quad Y \dashrightarrow \mathbb{P}^{n-1}$

$\quad y \longmapsto [1 : f_1(y) : \cdots : f_{n-1}(y)]$

Let Y' be the graph of this rational map

$$\begin{tikzcd} & Y' \arrow[dl, "\beta"'] \arrow[dr] & \\ Y & & \mathbb{P}^{n-1} \end{tikzcd}$$

regular map, it's dominant.

birational regular map $\quad \Rightarrow \beta^{-1}(\alpha(D))$ is a divisor. $\square$

Pf of Thm 7

Assume G is not complete. By induction.

$X := G$ as a G-torsor, Embed it

as an open subset of a complete var $\overline{X}$

$\bar{X} - X$ is $\begin{cases} \text{codim} \geq 2 \rightsquigarrow \text{blow up.} \\ \text{pure codim } 1. \end{cases}$

Then replace $\bar{X}$ by its normalization.

$G \times \bar{X} \dashrightarrow \bar{X}$ is defined on an open subset $U \subset G \times \bar{X}$ of codim ≥ 2
normal complet var

Take $E = \bar{X} \setminus X$, of pure codim 1 $\Rightarrow U \cap (G \times E)$ open ✓ dense in $G \times E$

We claim take $g \in G$, $x \in E$ $\Rightarrow$ if $g.x$ is def. $g.x \in E$

Else, $g.x \in X$, but $g^{-1}.(g.x)$ is def $= e.x$
$\qquad\qquad\qquad\qquad\quad \underset{X}{\cap}$ $\qquad\qquad\qquad \underset{x \in E}{\shortparallel}$ $\qquad \underset{E}{\shortparallel}$

Indeed we have an rational action $\alpha: G \times E \to E$

* Take E_1 an irr comp of E, apply the previous lemma to an open subset, we can get a birational map

$X' \longrightarrow X$ such that image of E_1 under

$G \times X \dashrightarrow X'$ is a divisor D.

And normalize X', replace X by X'

So setting the stage, we can assume there $\exists$ irr comp E_1' of E whose image D under $G \times X \dashrightarrow X$ is a divisor. Take $(g.x) \in V \underset{\text{open}}{\subset} G \times X$ where defined

$g.x$ is def, $x = h.y$ for $h \in G$, $y \in E_1'$

$\Rightarrow g.x = g.(h.y) = (gh).y \in D$

<u>Exercise</u>: $e.P = P$ is defined on $G \times D \dashrightarrow D$.

set $H' = \{g \in V \mid gP = P\}$ $\quad G \times G \dashrightarrow G$

define a rational $H' \times H' \dashrightarrow H'$

Take closure H of H', alg subgp of G

$\dim H \geq \dim G - \dim D = 1.$

① $H \neq G$. 1) H not complete, by induction,

contains an affine alg sub of dim > 0

2) H complete, $\exists\ \underline{N}$ which is complete in G

because $A \times N \to \underline{G}$

N contains an affine alg subgp of dim > 0

② $H = G$, G fixes P, it acts on $\mathcal{O}_P$

$$\rho_n : \underset{\underset{\text{affine}}{\downarrow}}{G} \longrightarrow GL(\mathcal{O}_P/m_P^{n+1}) \quad n >> 0.$$

$\square$

6) (Barsotti - Chevalley)

G gp var $| k$, There $\exists$ a (smallest) connected affine normal alg subgp N such that G/N is abelian.

If N is smooth, its formation will commute with base field.

If k is perfect, N is smooth.

Pf: smallest = unique minimal.

(*) there $\exists$ a connected affine normal alg subgp N in G such that G/N is abelian

$$G/N_1 \cap N_2 \hookrightarrow G/N_1 \times G/N_2$$

① The formation of base change

$k'|k$. $N \Rightarrow N_{k'}$ is also

$G_{k'}/N_{k'} \cong (G/N)_{k'}$ is abelian

N_n' may be not the smallest

But if we assume N is smooth.

② k alg closed. Prove by induction

Consider $Z = Z(G)$

We may assume $Z_{red} \neq 1$ (For else, $n \gg 0$
$G \to GL(\mathcal{O}_e/m_e^{n+1})$
with finite kernel $(= \# Z/Z_{red})$
$\Rightarrow G$ is affine)

i.) If Z_{red} is complete, N of Z_{red} in G

k is alg clo $\Rightarrow$ perfect $\Rightarrow$ N is smooth by N_{red}

$\Rightarrow \dim N < \dim G$, $\exists$ affine normal subgp var N_1 of N

such that N/N_1 is abelian N_1 is normal in G

G/N_1 is also abelian, because we have isogeny

$G/N_1 \leftarrow \cdots Z \times N/N_1$ induced from

$Z_{red} \times N \longrightarrow G$

ii) If Z_{red} is not complete, Rosenlicht's dichotomy. It contains an affine subgp var N

of $\dim > 0$. N is normal in G

$G/N \supseteq$ affine normal subvar N_1

s.t. $(G/N)/N_1$ is abelian

inverse N_1' of N_1 in G aff normal

$$(G/N)/N_1 \cong G'/N_1' \leftarrow \text{abelian}$$

□

3) If k is perfect, then apply Galois desent

theory $k \subseteq \boxed{k^{al}}$ N is def$/k$.

4) If k is not even perfect

$k'|k$ pure insep ext. $G' := G_{k'}$ contains an affine normal subgp N' s.t. G'/N' is abelian

Suppose $(k')^p \subseteq k$.

Frobenius: $F: G' \to (G')^{(p)} \left(:= G' \otimes_k k^{1/p} \right)$

Def N to be the pull back of $(N')^{1/p}$

consider $\mathcal{I}' \subset \mathcal{O}_{G'}$ of N'

$\mathcal{I} \subset \mathcal{O}_G$ is generated by p-th powers of local section $\mathcal{I}'$, but as we assume $k'^p \subseteq k \Rightarrow \mathcal{I}$ is just generated by local section of $\mathcal{O}_G$

$\Rightarrow N$ is defined over k. $\square$

Cor: Any p-ab var G over perfect field is abelian

Pf: N = trivial $\underset{\overset{\shortparallel}{G}}{G/N}$ is abelian

Cor: Any p-ab is commutative

Pf: $[G, G]$ is the smallest normal subgp of G

$G/\underline{[G,G]}$ is commutative.

Now G p-abelian, N G/N is abelian

$\Rightarrow [G,G] \underset{\text{closed}}{\subset} N \Rightarrow [G,G]$ is affine.

As $[G,G]$ is smooth affine connected normal

$\Rightarrow$ It's trivial $\square$