Semilinear Representations Over Some Rational Function Fields

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Representations

Definition of Smooth Rep

A smooth representation of a topological group G is a vector space V with a G-action and such that for any $v \in V$, the stabilizer of v is open in G.

Definition of Semilinear Rep

Let K be a field with a G-action (by field automorphisms), let $K\langle G\rangle$ be the group ring, namely, it consists of elements of the form $\sum_{i=1}^{n} a_i[g_i]$ with $a_i \in K$ and $g_i \in G$. The multiplication formula is given by $a[g] \cdot b[h] = ab^g[gh]$. A K-semilinear representation of G is just a left $K\langle G\rangle$-module.

Trivial Example

Take K to be a G-field. Let S be a G-set. Similarly, we can form the set $K[S]$, where G acts diagonally. It is then a K-semilinear representation of G.

Background

Let $G := \mathfrak{S}_\Psi$ be the symmetric group of an infinite set Ψ and K is a G-field. The main object we are interested is the category of smooth K-semilinear representations of G, denoted by $Sm_K(G)$. There are many interesting questions about this category, for example, to investigate the Gabriel spectrum of this category (for some specific field). skip slide

Goal

In particular, we are interested in the simple objects(i.e. irreducible representations) in $Sm_K(G)$. A natural question would be:

Question

Given any positive integer n, can we find some K such that there exists a simple object of dimension n in $Sm_K(G)$?

A first step to this question may be trying to construct examples of $2, 3, 4 \cdots$ dimensional examples of simple objects, with a choice of K.

Choice of K

The most natural G-field is $k(\Psi)$, the rational function field over the infinite set Ψ of variables. However, the category $Sm_{k(\Psi)}(G)$ only admits 1-dimensional simple objects. ⏩ skip slide

The next choice of K is some G-subfield of $k(\Psi)$. In particular, the Galois case is much easier.

Galois Case

Case (1)

$K := k(\Psi)^H$ where H is a finite subgroup of $Aut_G(k(\Psi)) \cong Aut(k(x)|k) \cong PGL(2,k)$ where x is any variable in Ψ.

Case (2)

$K := L^H$ where $k(\Psi)|L$ is a transcendental extension of G-fields and H is a finite subgroup of $Aut_G(L)$.

Method

Make use of the linear representation of H over k, for each irreducible representation ρ, we may form $V_\rho := Hom_{k[H]}(\rho, k(\Psi))$. It's an object in $Sm_K(G)$.

Theorem

V_ρ is irreducible and $\dim_K V_\rho = \dim_k \rho$.

Case (1)

The possible finite subgroups of $PGL(2, k)$ are rather limited. They are $\mathbb{Z}/r\mathbb{Z}$, D_r, A_4, S_4, A_5.
Cyclic groups only yield one-dimensional examples. D_r gives us (many) 2-dimensional examples. A_4 gives us 3-dimensional examples. S_4 is almost the same as A_4. A_5 will give $3, 4, 5$-dimensional examples.

Remark

Of course some assumptions need to be put on the base field k.

Case (2)

Theorem

If $char(k) = 0$ and L is algebraically closed in $k(\Psi)$, then the transcendence degree of $k(\Psi)|L$ is no more than 3. skip slide

We need to compute the automorphism group $Aut_G L$.

Case(2)

The possible automorphism groups are either $\mathbb{Z}/2\mathbb{Z}$ or trivial group or k^*, the multiplicative group of base field k. So it will only generate 1-dimensional examples.

Non-Galois Case

For example, $k(\Psi)^{A_4}|k(\Psi)^{A_5}$. For subfields K of $k(\Psi)$ over which $k(\Psi)$ is finite non-Galois, it turns out it's almost the same as the finite case:
Assume *char* $k = 0$, $K = L^H$ where $L = k(a)(\Psi)$ with $k(a)|k$ finite Galois extension. And H is a finite subgroup of $G(k(a)|k) \ltimes PGL(2, k(a))$

Extensions

We have considered the G-subfields of $k(\Psi)$. It's natural to consider G-extensions $L|k(\Psi)$ as well. However, if we require $L^G = k$, that the extension will preserve the fixed field, then there are almost no non-trivial examples. skip slide

Summary and Insight

Consider the G-subfields of $k(\Psi)$, we can construct $2, 3, 4, 5$-dimensional simple objects in $Sm_K(G)$. But they're all we can get in this way.
In general, it's plausible to construct more examples by considering the Cremona groups.

End

Thanks!

Theorem(M.Rovinsky)

Up to isomorphism of $\mathfrak{G}_\Psi$-fields,
(1) If $tr.deg(k(\Psi)|K) = 3$, then
$K = k(\frac{x-y}{x-w}\frac{z-w}{z-y}|x, y, z, w$ pairwise different in $\Psi)$;
(2) If $tr.deg(k(\Psi)|K) = 2$, then
$K = k(\frac{x-y}{x-z}|x, y, z$ pairwise different in $\Psi)$;
(3) If $tr.deg(k(\Psi)|K) = 1$, then either $K_1 = k(x - y|x, y \in \Psi)$ or
$K_2 = k(\frac{x}{y}|x, y \in \Psi)$. skip slide

Gabriel Spectrum

The Gabriel spectrum S of a grothendieck category $\mathfrak{C}$ is a topological space. The points of the space consists of isomorphism classes of indecomposable injectives in $\mathfrak{C}$. For any $X \in \mathfrak{C}$, set $[X] := \{E \in S | Hom(X, E) = 0\}$. Take $[X]$ as closed subsets of S.

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Take R to be a noetherian unital commutative ring and $\mathfrak{M}$ is the category of modules over R. Then the Gabriel spectrum of $\mathfrak{M}$ is homeomorphic to the usual prime spectrum $Spec(R)$.

Theorems

Theorem 1(M. Rovinsky)

Let K be a G-subfield of $k(\Psi)$, then any smooth K-semilinear representation of G can be embedded into a direct product of copies of $k(\Psi)$; any smooth $k(\Psi)$-semilinear representation of G of finite length is isomorphic to a direct sum of copies of $k(\Psi)$.

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Theorem 2(M. Rovinsky)

Let Y be a geometrically irreducible k-variety. If we have a G-extension $k(\Psi)(Y)|k(\Psi)$, then there exists a geometrically irreducible k-variety Y' such that $(k(\Psi)(Y))^G \cong k(Y')$. Moreover, $Y_{k'}$ is birational to $Y'_{k'}$ for a finite field extension $k'|k$.

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